Dihedral Groups

(Section 2.3)

Recall: (symmetries of the square and equilateral triangle)

We denoted by D_3 the symmetries of the equilateral triangle, and by D_4 the symmetries of the square.

We will extend this notation to all regular polygons!

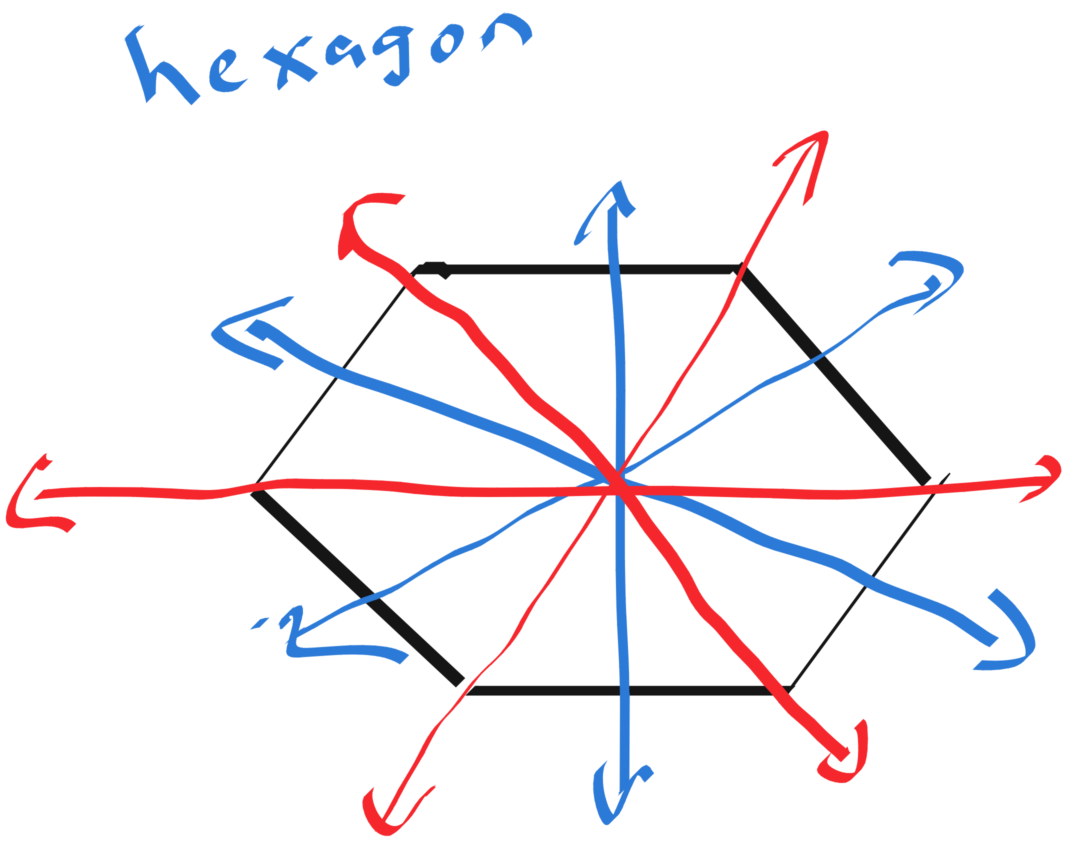
Definition: (finite dihedral groups) Let P be a regular n-sided polygon. We denote by D_n the group of all symmetries of P. D_n is the **dihedral group of order $2n$**.

Q: Why $2n$ instead of n?

A: An n-sided regular polygon has rotational symmetries e (do nothing) and powers of rotation by $\frac{2\pi}{n}$ radians, denoted $R_{\frac{2\pi}{n}}$.

Then there are n flips, characterized as follows:

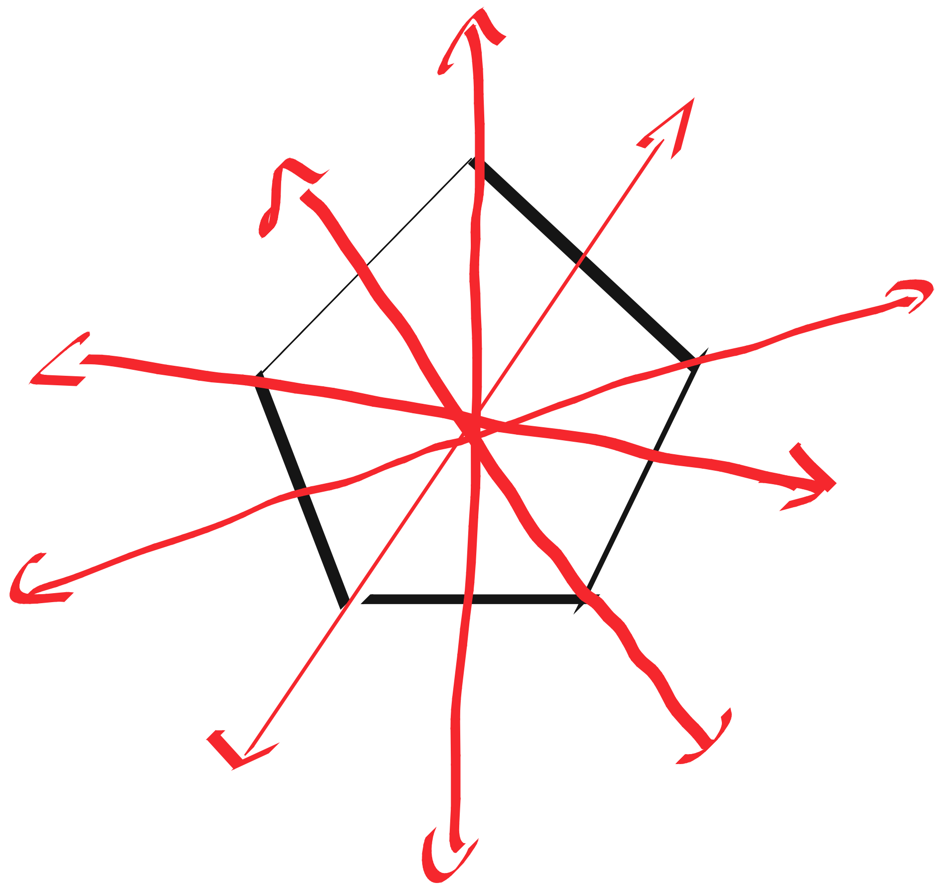
If n is even there are $\frac{n}{2}$ flips corresponding to pairing opposite vertices and an additional $\frac{n}{2}$ flips from pairing opposite sides



red lines = flips about vertices

blue lines = flips about sides

If n is odd there are n flips corresponding to pairing a vertex with an opposing side



Theorem: (Characterization of finite dihedral groups) Every element of D_n can be expressed as

$$\boxed{R^k J^\ell}$$

where R is a $\frac{2\pi}{n}$ rotation and J is a choice of flip, with $0 \le k < n$, $\ell \in \{0,1\}$ with the convention that $R^0 = J^0 =$ do nothing.

Moreover,

$$\boxed{RJ = JR^{n-1}}$$

$\ell = 0$ We get rotations

$\ell = 1$ We get flips

Q: What is the group of symmetries of a circle?

A: The symmetries are either rotations or flips about a diameter of the circle. You can rotate through any angle (modulo 2π) and any diameter, so the order is infinite and the group is of uncountable cardinality since $[0, 2\pi)$ is of uncountable cardinality

countable = bijective correspondence with $\mathbb{N}$
(countably infinite)

uncountable = infinite, no bijection with $\mathbb{N}$

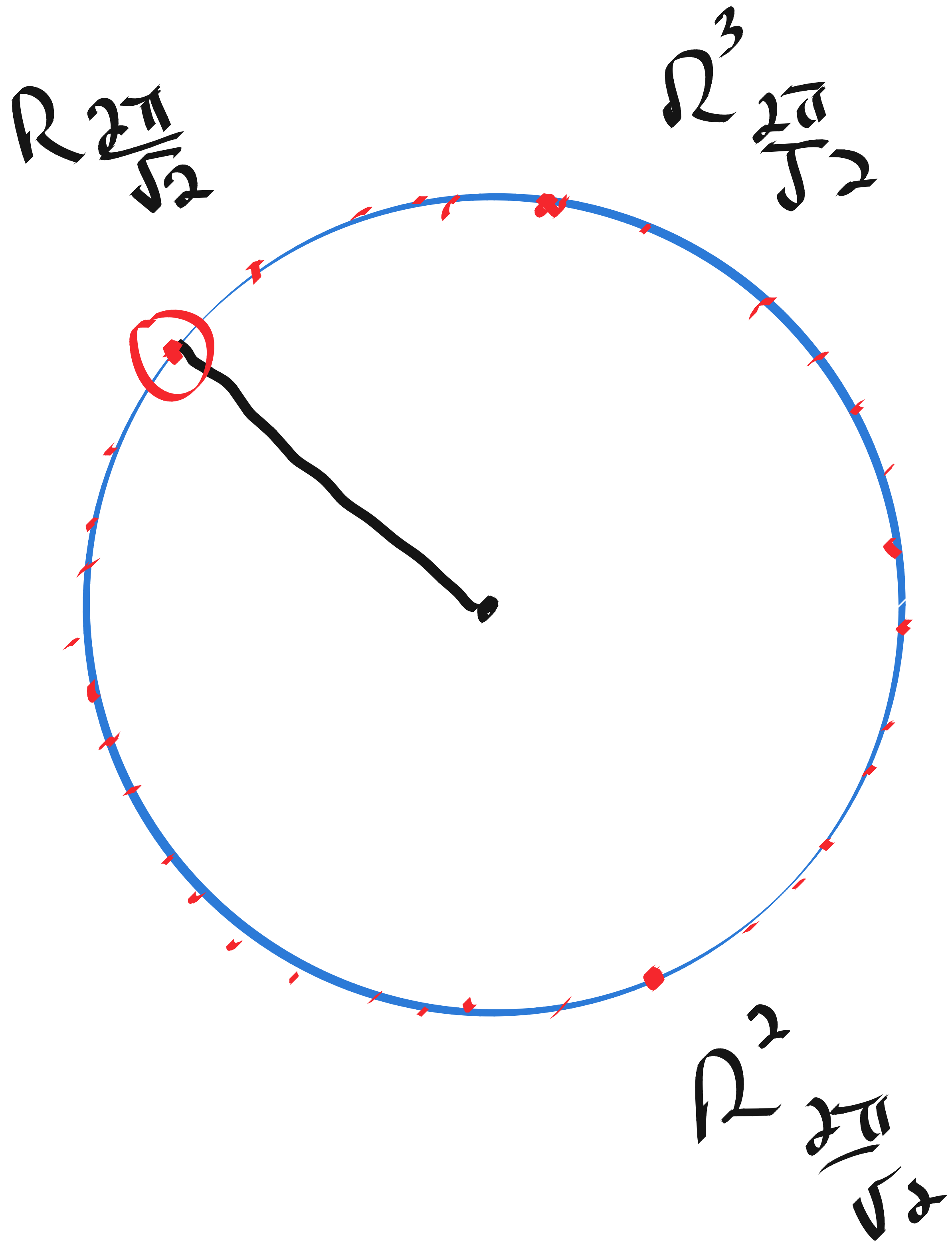
No flip is a rotation since flips fix precicesly two points on the circle, and (nonidentity) rotations fix no points!

However, we still have the relation

$$RJ = JR^{-1}$$

where R is any rotation and J is any flip.

Note: if R rotates by $\frac{2\pi}{n}$, then R has finite order. But if R rotates by an irrational multiple of 2π, then R has infinite order!



Why is the order infinite?

If $\exists$ $m \in \mathbb{N}$

$$R^m_{\frac{2\pi}{\sqrt{2}}} = e,$$

then

$$R^m_{\frac{2\pi}{\sqrt{2}}} = R^m_{\frac{2\pi}{m}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{1}{m}$$

$$\Rightarrow \sqrt{2} = m, \text{ impossible.}$$

A similar result occurs with choosing rational multiples of 2π - $\frac{1}{\sqrt{2}}$ is not rational!